# STABILITY OF NONLINEAR LOCAL AXISYMMETRICAL STRAINS OF SHELLS OF REVOLUTION

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A brief review of the results of investigation of the stability of the axisymmetrical strains of elastic shells of revolution is contained in [1, 2]. In [3] the problem was formulated and solved for a round shell, uniformly loaded along its hinged edge by a radial compressive force. Below, this problem is formulated for an arbitrary shell of revolution with a uniformly compressed hinged edge. Results of its solution are given for conical and spherical shells.

§1. We consider an arbitrary shell of revolution, closed in a peripheral direction, and having at least one edge in an axial direction. The following variable quantities are introduced in an arbitrary cross section of the middle surface of the shell (Fig. 1): $x_1$ is the coordinate in the direction of the meridian; $l_1$ is the Lamé parameter of this direction; $\theta$ is the angle between the axis of rotation and a normal to the meridian; $l_2$ is the radius of a parallel circle. The position of the axial cross section is determined by the peripheral coordinate $x_2$, which is identified with the solar angle in the plane of the parallel circle, so that $l_2$ is the Lamé parameter of the axial direction. The symbols $k_1$ and $k_2$ denote the normal curvatures of the coordinate axes $x_1$ and $x_2$.

The following is postulated: a) the shell is uniformly compressed along one edge in such a way that the peripheral deformation at the edge $\varepsilon = \text{const} < 0$ is known; b) the compressed edge is free with respect to meridional rotation and axial displacement and attached with respect to peripheral displacement; c) the shell has a constant thickness h and is made of a linearly elastic homogeneous material with the Young modulus E and the Poisson coefficient $\nu$.

Let the radius of the compressed edge be equal to b, and the angle of inclination of a normal to the axis of rotation at it be equal to $\beta$.

The coordinates $x_1$ and $x_2$ are the principal coordinates of the strained middle surface. They are taken as independent variables. The origin of the coordinate $x_1$ is located at the compressed edge (the direction of reckoning corresponding to Fig. 1 is negative), and of the coordinate $x_2$ at an arbitrarily selected meridian (direction of reckoning not significant). These same coordinates are used as Lagrangian coordinates of points of the deformed surface.

The following notation is introduced for the stress–strain state of the shell: $u_1$, $u_2$, w are the displacements of a point of the middle surface in meridional, peripheral, and normal directions, respectively;

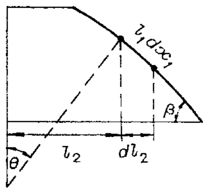


Fig. 1

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$\varepsilon_{ij}$ are the components of the tangential strain; $\varkappa_{ij}$ are the components of the bending strain; $N_{ij}$ are the tangential stresses; $Q_i$ are the intersecting stresses; P is the axial stress; $M_{ij}$ are the specific moments (the latter subscripts run through the values 1 and 2).

The subcritical stress–strain state of the shell under consideration will be axisymmetrical and localized in the neighborhood of the compressed edge. Its kinematic and stress characteristics are denoted by the index °.

If we introduce the constants $A = 1/Eh$; $D = Eh^3/12(1-\nu^2)$; $B = \sqrt{D/A}$; $C = \sqrt{AD}$, then, in correspondence with assumption c), the equations of the connection between the stress characteristics and the components of the axisymmetrical strain are written in the form

$$\mathring{\varepsilon}_{22} = A\left(\mathring{N}_{22} - \nu \mathring{N}_{11}\right);\quad \mathring{M}_{11} = D\left(\mathring{\varkappa}_{11} + \nu\mathring{\varkappa}_{22}\right),\quad 1 \rightleftarrows 2. \tag{1.1}$$

Assuming that the axisymmetrical deformation of a shell with local compression can be accompanied by strong bending of the meridian of the middle surface, to describe it we use the nonlinear equations of E. Reissner [4]. To this end we introduce the angle of rotation $v(x_1)$ of the element $l_1 dx_1$ during the process of axisymmetrical deformation and the stress function $\eta(x_1)$, with an accuracy up to a constant factor B, coinciding with the corresponding function of E. Reissner.

In the deformed position of an element, its normal will form with the axis of rotation an angle equal to $\theta^0(x_1) = \theta(x_1) + v(x_1)$. The components of the strain are expressed in terms of the function v using the formulas (primes denote differentiation with respect to $x_1$)

$$l_1\mathring{\varkappa}_{11} = v',\quad l_2\mathring{\varkappa}_{22} = \sin\theta^\circ - \sin\theta.$$

Thanks to the absence of surface loads, the formulas of E. Reissner, expressing the stresses in terms of the functions $\eta$ and $\theta^\circ$, are simplified to the form

$$l_1\mathring{N}_{22} = B\eta',\quad l_2\mathring{N}_{11} = B\eta\cos\theta^\circ,\quad l_2\mathring{Q}_1 = B\eta\sin\theta^\circ.$$

The equations of Reissner establish a nonlinear connection between the functions $\eta$ and v. In the present case, they describe the local strain in the neighborhood of the compressed edge of the shell and can be simplified in accordance with the asymptotic theory of the edge effect. To this end, we assume that

$$l_1 = \sqrt{2Cb},\quad l_2(x_1) \simeq b,\quad \theta(x_1) \simeq \beta$$

and, in the starting equations, we discard terms with the natural small parameter,

$$\mu = l_1/b = \sqrt{2C/b} \ll 1. \tag{1.2}$$

As a result, we arrive at the following system of nonlinear equations with respect to the functions $\eta$ and v:

$$\begin{aligned} &\eta'' + 2\sin\beta\sin v + \cos\beta(1-\cos v)] = 0;\\ &v'' - 2\eta(\sin\beta\cos v + \cos\beta\sin v) = 0. \end{aligned} \tag{1.3}$$

In accordance with assumptions a), b), the boundary conditions at the edge $x_1 = 0$ are formulated in the form

$$\mathring{\varepsilon}_{22}(0) = \varepsilon;\quad \mathring{M}_{11}(0) = 0.$$

After appropriate simplifications, and after introduction of the parameter $p = -\varepsilon/\mu$, these conditions assume the form

$$\eta'(0) = -2p;\quad v(0) = 0. \tag{1.4}$$

To obtain a solution of the system (1.3) which is damped with increasing distance from the edge, we require satisfaction of the following conditions:

$$\eta(-\infty) = 0;\quad v(-\infty) = 0. \tag{1.5}$$

Equations (1.3), which can be called the equations of the nonlinear edge effect, along with the boundary conditions (1.4), (1.5), describe the nonlinear local axisymmetrical strain of an arbitrary shell of revolution, uniformly compressed along a hinged edge.

It must be noted that, for such a simplified formulation of the problem, the satisfaction of condition (1.2) is not sufficient, since a transition to the system (1.3) is possible with the supplementary limitation

$$|\sin\theta^\circ| \gg \mu, \tag{1.6}$$

and the replacement of the real boundary conditions at the edge which is free of a load, by conditions (1.5), is possible if the distance between the edges considerably exceeds the value $l_1 = \sqrt{2Cb}$.

On the contrary, the requirement of constancy of the thickness of the shell is superfluous. It is sufficient to require that the variability of the thickness not exceed the variability of the functions $l_2$ and $\theta$. Then, without increasing the error, the constant h can be identified with the thickness of the shell at the compressed edge.

§2. The possibility of instability of the axisymmetrical strain of a shell is connected with the presence of mixed unsymmetrical forms of equilibrium at the shell. For the increments taken on by the kinematic and stress characteristics of the strain with a transition from an axisymmetrical form of the equilibrium to a nonaxisymmetrical form, we adopt the notation of Sec. 1. These increments are functions of the coordinates $x_1$ and $x_2$; the dependence on the second coordinate is periodic.

Taking account of the local character of the nonaxisymmetrical strain, the system of equations of the stability can be taken in a simplified form, in accordance with the theory of inclined shells.

If we introduce the operators $(y = l_2\ l_1')$

$$L_1 = \frac{1}{l_1^2}\frac{\partial^2}{\partial x_1^2}, \quad L_2 = \frac{1}{l_2^2}\left(\frac{\partial^2}{\partial x_2^2} + y\cos\theta\frac{\partial}{\partial x_1}\right),$$
$$\nabla^2 = L_1 + L_2, \quad L = \frac{1}{l_1 l_2}\left(\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\cos\theta}{y}\frac{\partial}{\partial x_2}\right),$$

the function v $(x_1, x_2)$, determining the increments of the tangential stresses in the form

$$N_{22} = -BL_1 v, \ N_{11} = -BL_2 v, \ N_{21} = N_{12} = BLv, \tag{2.1}$$

and use the known formulas of the theory of inclined shells

$$\varkappa_{11} = -L_1 w, \quad \varkappa_{22} = -L_2 w, \quad \varkappa_{12} = \varkappa_{21} = -Lw; \tag{2.2}$$
$$Q_1 = -\frac{D}{l_1}\frac{\partial}{\partial x_1}\nabla^2 w, \quad Q_2 = -\frac{D}{l_2}\frac{\partial}{\partial x_2}\nabla^2 w,$$

then the system of equation of the stability of the axisymmetrical equilibrium state is written in the form

$$C\nabla^2\nabla^2 v + \left(k_2 + \overset{\circ}{\varkappa}_{22}\right)L_1 w + \left(k_1 + \overset{\circ}{\varkappa}_{11}\right)L_2 w = 0,$$
$$C\nabla^2\nabla^2 w - \left(k_2 + \overset{\circ}{\varkappa}_{22}\right)L_1 v - \left(k_1 + \overset{\circ}{\varkappa}_{11}\right)L_2 v - B^{-1}\left(\overset{\circ}{N}_{11}L_1 w + \overset{\circ}{N}_{22}L_2 w\right) = 0.$$

After simplifications such as were made in Sec. 1, for the amplitude of $v_n(x_1)$ and $w_n(x_1)$ periodic along the coordinate $x_2$ of the solution, we obtain the following system of two ordinary differential equations:

$$v_n^{IV} - 2\gamma^2 v_n'' + \gamma^4 v_n + 2\left[bk_{20}w_n'' - (bk_{10} + qv'/p)\gamma^2 w_n\right] = 0; \tag{2.3}$$
$$w_n^{IV} - 2\gamma^2 w_n'' + \gamma^4 w_n - 2\left[bk_{20}v_n'' - (bk_{10} + qv'/p)\gamma^2 v_n\right] + 2q(\eta'/p)\gamma^2 w_n = 0.$$

Here n is the number of waves in a peripheral direction;

$$q = -\varepsilon/\mu^2; \ \gamma = \mu n; \ k_{i_0} = k_i(0). \tag{2.4}$$

The boundary conditions for the increments, corresponding to assumptions a), b), are homogeneous and have the form

$$\varepsilon_{22}(0) = 0; \ M_{11}(0) = 0; \ P(0) = 0; \ u_2(0) = 0. \tag{2.5}$$

For nonaxisymmetrical perturbations being considered here, the connecting equations of the form (1.1) must be supplemented by the following equations, valid within the framework of the theory of inclined shells:

$$\varepsilon_{21} = (1+\nu)AT_{21}; \ M_{12} = (1-\nu)D\varkappa_{12}, \ 1 \rightleftarrows 2.$$

Using the connecting equations and formulas (2.1), (2.2), the first two of equalities (2.5) are expressed in terms of the functions v and w, adopting, for their amplitude, the simplified form

$$v_n'' + \nu\gamma^2 v_n = 0; \quad w_n'' - \nu\gamma^2 w_n = 0,$$

and the equality

$$P \equiv N_{11}\sin\theta^\circ - \left(Q_1 + \frac{1}{l_2}\frac{\partial M_{12}}{\partial x_2}\right)\cos\theta^\circ = 0$$

is brought to the form

$$\gamma^2 v_n \sin\theta^\circ + \frac{1}{2}\mu\left[w_n''' - (2-\nu)\gamma^2 w_n'\right]\cos\theta^\circ = 0.$$

With satisfaction of the condition

$$|\cos\theta^\circ| \leqslant |\sin\theta^\circ| \tag{2.6}$$

the equality $P=0$, with the adopted degree of accuracy, corresponds to the equality $v_n=0$. It can be shown in a similar way that, under these circumstances, the equality $u_2$ corresponds to the equality $w_n=0$. Consequently, with the limitation (2.6), the equalities (2.5) give the following boundary conditions at the point $x_1=0$ for the system (2.3):

$$v_n(0)=0;\quad w_n(0)=0;\quad v_n''(0)=0;\quad w_n''(0)=0. \tag{2.7}$$

At infinity, we require the satisfaction of the damping conditions

$$v_n(-\infty)=0;\quad w_n(-\infty)=0;\quad v_n'(-\infty)=0;\quad w_n'(-\infty)=0. \tag{2.8}$$

We note that, with condition (2.6), condition (1.6) is also satisfied; therefore, we can concern ourselves only with the satisfaction of the first of them.

Equations (2.3) with the boundary conditions (2.7), (2.8) formulate the problem of the stability of the axisymmetrical stress–strain state of the equilibrium of an arbitrary shell of revolution, uniformly compressed along a hinged edge.

The solution of this problem depends on three generalized parameters: $\gamma$, p, q. As can be seen from (2.4), the parameter q can determine the value of the forced strain of the compressed edge. In this case, the parameter p characterizes the thinness of the wall of the shell, since $p=\mu q$, where $\mu$ is the parameter of the thinness of the wall (1.2). The parameter $\gamma$ must be understood as the parameter of the wave formation, since $\gamma=\mu n$, where n is the number of peripheral waves, formed with a loss of the stability of the shell.

With such an interpretation of the parameters, it becomes evident that the determination of the critical value of the forced peripheral strain of the edge of the shell reduces to the determination, with a given value of the parameter p, of the smallest eigenvalue of the parameter q with respect to $\gamma$, from the homogeneous boundary-value problem for the system (2.3).

Starting from equalities (1.3)-(1.5), it can be established that, with $p\to 0$, the ratios $\eta'/p$ and $v'/p$ figuring in Eqs. (2.3) tend toward values corresponding to the linear boundary-value problem of the axisymmetrical bending of a shell. Consequently, with $p=0$, the problem of the eigenvalues of (2.3), (2.7), (2.8) determines the limit of the stability of a linear axisymmetrical equilibrium state, taking account of its moment character. But since from the equality $p=0$ there flows the equality $\mu=0$ (since $\varepsilon\neq 0$), we arrive at the conclusion that the linear statement of the problem is correct only for an infinitely thin shell.

§3. The process of the solution of the problem posed, of the stability of the axisymmetrical strain of a shell of revolution, is divided into two consecutive stages. The first stage is the solution of the inhomogeneous boundary-value problem posed in Sec. 1, describing the nonlinear local axisymmetric strain of a shell. The second stage is solution of the problem of eigenvalues posed in Sec. 2.

The method used for numerical solution of these boundary-value problems is described in [3]. The purpose of the numerical experiment was to investigate the dependence of the critical value of the forced strain on the geometric parameters of that part of the shell which is adjacent to the compressed edge. To this end, stability calculations were made of conical and spherical shells, with the following three values of the angle $\beta$:

$$\beta_1=\pi/2,\ \beta_2=\pi/3,\ \beta_3=\pi/6. \tag{3.1}$$

For every shell with a given geometry of the middle surface, the dependence of the critical value $q^*$ of the parameter q on p, i.e., in the final analysis, on $\mu$, was determined. This dependence can be symbolically represented in the form $q^*=q^*(p)=q^*(\mu)$.

Calculating for $p=0$, values of $q^*(0)=q^+$, and the corresponding values $\gamma^+$ of the parameter $\gamma$, are given in Table 1. The number of the column indicates which of the values (3.1) of the angle $\beta$ corresponds to the numbers in these columns.

In accordance with the content of the last paragraph of Sec. 2, the values of $q^+$ in Table 1 can be understood as critical for the linear axisymmetrical strain of a shell. The effect of the nonlinear char-

TABLE 1

| Parameters | Conical shells | | | Spherical shells | | |
|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 1 | 2 | 3 |
| $q^+$ | 32,7 | 30,2 | 23,3 | 35,1 | 32,7 | 25,2 |
| $\gamma^+$ | 4,9 | 4,55 | 3,45 | 5,1 | 4,75 | 3,65 |
| $\gamma^-$ | 2,5 | 2,1 | 1,35 | 2,6 | 2,2 | 1,4 |

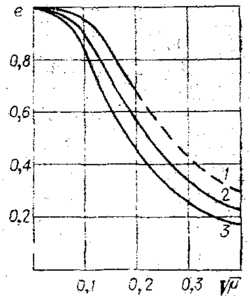


Fig. 2

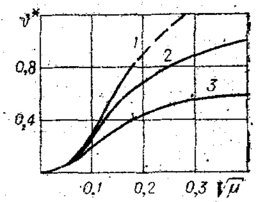


Fig. 3

acter of this strain on the value of $q^*$ is reflected in Figs. 2 and 3, using the graphical dependence of the ratio $e = q^*/q^+$ on $\sqrt{\mu}$. The values of this ratio practically coincide for conical and spherical shells, with an identical angle $\beta$. Each of the values of the angle $\beta$ (3.1) corresponds to the curve with the corresponding number. The change in the values $\gamma^*$ of the parameter $\gamma$ corresponding to $q^*$ with a rise in $\mu$ can be judged from the number $\gamma^-$ given in the last row of Table 1, which determine the values of $\gamma^*$ with $\sqrt{\mu} = 0.4$. With a change in the value of $\sqrt{\mu}$ from 0 to 0.4, the values of $\gamma^*$ vary from $\gamma^+$ to $\gamma^-$.

The dependence on $\sqrt{\mu}$ of the critical values $v^*$ of the angle of rotation at the compressed edge is illustrated graphically in Fig. 3 (for conical and spherical shells, the curves coincide). The values of $v^*$ which correspond to the dashed segments of the curves do not satisfy condition (2.6). Consequently, for shells with an angle $\beta = \pi/2$, it cannot be guaranteed that, in the region $0.2 < \sqrt{\mu} \leq 0.4$, the solution of the problem (2.3), (2.7), (2.8) will satisfy the initial boundary conditions (2.5) with the required degree of accuracy. It must be regarded as a formal solution. The dashed segments of the curves in Fig. 2 have the same formal meaning.

The critical values of the forced (maximal) strain $\varepsilon^*$ and the maximal stress $\sigma^*$ were calculated using the formulas $\varepsilon^* = -eq^+\mu^2$; $\sigma^* = E\varepsilon^*$.

For the shells calculated, the values of the parameter $q^+$ are determined using Table 1, and the values of the coefficient e using the graphical dependences illustrated in Fig. 2.

## LITERATURE CITED


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